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Stringy effect of the holographic correspondence for Dp-brane backgrounds

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Abstract: Based on the holographic conjecture for superstrings on Dp-brane backgrounds and the dual $(p+1)$-dimensional gauge theory $(0 \leq p \leq 4)$ given in [1] and [2], we continue the study of superstring amplitudes including string higher modes $(n \neq 0)$. We give a prediction to the two-point functions of operators with large R-charge J. The effect of stringy modes do not appear as the form of anomalous dimensions except for $p = 3$. Instead, it gives non-trivial correction to the two-point functions for supergravity modes. For $p = 4$, the scalar two-point functions for any n behave like free fields of the effective dimension $d_{\text{eff}} = 6$ in the infra-red limit.

Keywords: Superstrings and Heterotic Strings, D-branes, AdS-CFT and dS-CFT Correspondence.

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A. Diagonalization of S-matrix from fermionic oscillators **20**

1. Introduction

The BMN conjecture [3] gives an important step toward the realization of the AdS/CFT correspondence [4] including stringy effect. Originally, AdS/CFT correspondence states that there is a holographic relation between bulk supergravity theory in the $\mathrm{AdS}_5 \times S^5$ geometry and the boundary $\mathcal{N} = 4$ U(N) super Yang-Mills theory. The relation between bulk field $\phi(x_\mu, z)$ and the corresponding boundary operator $\mathcal{O}(x_\mu)$ is given explicitly by the GKP/Witten relation [5, 6]. On the other hand, BMN conjecture, which is a relation between the superstring amplitudes in the plane-wave limit of $\mathrm{AdS}_5 \times S^5$ geometry and the $\mathcal{N} = 4$ SYM theory, does not seem to have such a holographic interpretation in its original form. In ref. [7], an interpretation of the BMN conjecture as a holographic relation of GKP/Witten type is presented and it is further studied to construct a consistent string field theory [8].

Other than the conventional $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, we can expect that such string/gravity correspondence can be applied to non-conformal case, such as the relation between Dp-brane background and the $(p+1)$-dimensional U(N) SYM theory with 16 supercharges [9]. For $p = 0$, holographic correspondence between supergravity theory and the 1-dimensional SYM was studied [10, 11] and the correspondence between supergravity states and SYM operators is given based on the generalized conformal symmetry [12].

In our previous papers with Y. Sekino and T. Yoneya ref. [1, 2], extending the idea given in [7], we investigated the closed superstring action around the null geodesic of Dp-brane backgrounds and calculated the 'diagonal' form of the boundary-to-boundary S-matrix operator. Our claim is that such S-matrix operator gives the two-point functions of the boundary gauge theory, which leads to the result consistent with the BMN conjecture. We mainly considered supergravity modes and obtained the two-point functions of certain operators $\mathcal{O}$ with large R-charge J for boundary $(p+1)$-dimensional gauge theory and found that the result is consistent with the field theory analysis using the supergravity theory: The two-point functions for supergravity modes show the power-law behavior and the contribution from the zero-point energies, which remain non-zero for $p \neq 3$, precisely agree with the supergravity results that are relevant for any J.

Our aim of the present paper is to study the effect of string higher modes on the S-matrix and to investigate the properties of two-point functions of the boundary gauge theory predicted by the S-matrix. We analyze the effect of $|n| \neq 0$ modes perturbatively around $|n| \to 0$ or $|n| \to \infty$. Then the UV or IR behavior of the two-point functions of dual gauge theory can be extracted. In particular, for $p = 4$, the scalar two-point functions for any n behave as for free fields in the effective dimension $d_{\text{eff}} = 6$ in the infra-red limit, which suggests the existence of non-trivial fixed points.

This paper is organized as follows. In the next section 2, we review the superstring dynamics around the (tunneling) null geodesic for near horizon limit of Dp-brane geometry. Then we set up our problem to analyze, i.e., we explicitly write down the quadratic hamiltonian $H = H_{\text{b}} + H_{\text{f}}$ representing the superstring fluctuations around the geodesic. In section 3, we perform the quantization of the system and give the diagonalized form of S-matrix operator. In section 4, after reviewing the procedure that gives the two-point functions $\langle \bar{\mathcal{O}}(x_f)\mathcal{O}(x_i)\rangle$ in terms of the S-matrix and the result for supergravity modes $n = 0$, we analyze the effect of string higher modes $n \neq 0$ on the two-point functions. In section 4.1, we give the perturbative analysis of the S-matrix with respect to small $|n|L^2/J$ and in section 4.2, we give the result for $|n|L^2/J \to \infty$ limit. Finally in section 4.3, we give the interpretation of the above result as the correlation functions of the $(p+1)$-dimensional gauge theory. In section 5, we give some concluding remarks. In the appendix, we explain the diagonalization procedure for the fermionic part of the S-matrix operator.

2. Superstring dynamics around the null geodesic in Dp-brane backgrounds

Dp-brane background for type II supergravity in the near-horizon limit is represented by the metric, Ramond-Ramond $(p+2)$-form field strength and the dilaton as

$$
\begin{aligned}
ds^2 &= \left[H^{-1/2}(-dt^2 + dx_a^2) + H^{1/2}(dr^2 + r^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\Omega_{6-p}^2))\right]\Big|_{\text{orig.}}, \\
F_{p+2} &= \left[\partial_r H^{-1} dt \wedge dx_1 \wedge \cdots \wedge dx_p \wedge dr\right]\Big|_{\text{orig.}}, \\
e^{\phi} &= g_s H^{\frac{3-p}{4}}, \qquad H = \frac{q_p}{r^{7-p}}\Big|_{\text{orig.}},
\end{aligned}
\tag{2.1}
$$

where $a = 1, \ldots, p$ and $q_p = \tilde{c}_p g_s N \ell_s^{7-p}$ with $\tilde{c}_p = 2^{5-p}\pi^{(5-p)/2}\Gamma(\frac{7-p}{2})$. We demand that the coupling and curvature radius are small ($e^\phi \ll 1$ and $rH^{\frac{1}{4}}/\ell_s \gg 1$) in order to suppress higher order corrections in α' [9]. If we take $N \to \infty$ with fixed large $g_s N$, the condition is satisfied in the whole near-horizon region except at the origin $r = 0$. (In our analysis, we only use the information far from $r = 0$.) We will obtain information of the dual gauge theory (with $g_s \sim g_{\rm YM}^2 \ell_s^{3-p}$) from the analysis of superstring theory on this background.

We will define dimensionless coordinates by rescaling the original coordinates given in (2.1) as $(t, x_a, r)|_{\rm orig.} \to L(t, x_a, r)$ with $L = q_p^{1/(7-p)}$. In the new coordinates, the equations in (2.1) are represented as

$$\begin{aligned} ds^2 &= L^2 \left[H^{-1/2}(-dt^2 + dx_a^2) + H^{1/2}(dr^2 + r^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\Omega_{6-p}^2)) \right], \\ F_{p+2} &= L^{p+1}\partial_r H^{-1} dt \wedge dx_1 \wedge \cdots \wedge dx_p \wedge dr\,, \\ e^\phi &= g_s H^{\frac{3-p}{4}}\,, \qquad H = \frac{1}{r^{7-p}}\,. \end{aligned} \tag{2.2}$$

In the following, we will use these dimensionless coordinates.

Note that this background (or the superstring action $S = S_{\rm b} + S_{\rm f}$ defined on the background) is invariant under the generalized scaling transformation

$$L \to \lambda^{\frac{3-p}{7-p}} L\,, \qquad (t, x_a) \to \lambda^{-\frac{2(5-p)}{7-p}}(t, x_a)\,, \qquad r \to \lambda^{\frac{4}{7-p}} r\,. \tag{2.3}$$

In the original coordinates, this transformation is represented as [12, 1] $g_s \to \lambda^{3-p} g_s$, $(t, x_a)|_{\rm orig.} \to \lambda^{-1}(t, x_a)|_{\rm orig.}$ and $r_{\rm orig.} \to \lambda r_{\rm orig.}$.

We consider the Green-Schwarz superstring action in the above background after performing 'double Wick rotation' [7, 1] for time and angle as $t \to -it$ $\psi \to -i\psi$. The bosonic part of the action is

$$S_{\rm b} = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi\tilde{\alpha}} d\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \tilde{g}_{\mu\nu}\,, \tag{2.4}$$

where α, β denote the world-sheet coordinates τ, σ with signature $(+,+)$ and $\tilde{\alpha}$ denotes the world-sheet length scale which will be fixed later. The metric $\tilde{g}_{\mu\nu}$ is given by (2.2) after the double Wick rotation. We set $\alpha' (= \ell_s^2) = 1$ hereafter.

We consider a point-like classical solution of the above action with $x_a = \theta = 0$, which has conserved energy and angular momentum along ψ:

$$E = L^2 r^{\frac{7-p}{2}} \dot{t} \sqrt{h} h^{\tau\tau} \tilde{\alpha}\,, \qquad J = L^2 r^{-\frac{3-p}{2}} \dot{\psi} \sqrt{h} h^{\tau\tau} \tilde{\alpha}\,. \tag{2.5}$$

By choosing the gauge for the world-sheet metric and $\tilde{\alpha}$ as

$$\sqrt{h} h^{\tau\tau} = \left(\frac{\cosh\tau}{\ell} \right)^{(3-p)/(5-p)}\,, \qquad \tilde{\alpha} = \frac{5-p}{2} \frac{J}{L^2} \tag{2.6}$$

the solution is written as

$$z = \frac{\tilde{\ell}}{\cosh\tau}\,, \qquad t = \tilde{\ell} \tanh\tau\,, \qquad \psi = \frac{2}{5-p}\tau\,, \tag{2.7}$$

where $z = \frac{2}{5-p} r^{-(5-p)/2}$, $\ell \equiv J/E$ and $\tilde{\ell} = \frac{2}{5-p}\ell$. For $0 \le p \le 4$, this solution represents the null geodesic connecting two points $t = t_i$ and $t = t_f$ on the $(p+1)$-dimensional boundary[1] $z \sim 0$ of $(p+2)$-dimensional space (t, z, x_a) which is conformal to AdS_{p+2}. The separation between the two end-points at $z(= 1/\Lambda) \sim 0$ is $|t_f - t_i| \sim 2\tilde{\ell}$. We sometimes call this geodesic 'tunneling null geodesic.' In our previous two papers [1, 2] with Y. Sekino and T. Yoneya, we claimed that the two-point functions $\langle \mathcal{O}(t_i)\mathcal{O}(t_f)\rangle$ of BMN type operators $\mathcal{O}$ can be obtained from the investigation of various modes of the string fluctuations around the geodesic.

To illustrate this discussion explicitly, we first present the quadratic action for fluctuations around the geodesic. It is given by expanding the Green-Schwarz action around the classical solution $\Xi = \bar{\Xi}(\tau)$ as

$$\Xi(\tau,\sigma) = \bar{\Xi}(\tau) + \frac{\Xi^{(1)}(\tau,\sigma)}{L} + \frac{\Xi^{(2)}(\tau,\sigma)}{L^2} + \cdots ,$$

where Ξ is each bosonic or fermionic field appearing in the action. By taking the gauge (2.6), the resulting bosonic action up to quadratic order is

$$S_{\rm b}^{(2)} = \frac{1}{4\pi}\int d\tau \int_0^{2\pi\tilde{\alpha}} d\sigma \left[\left(\frac{2}{5-p}\right)^2 (\dot{x}_i^2 + \bar{r}(\tau)^{p-3} x_i'^2) + \dot{y}_l^2 + \bar{r}(\tau)^{p-3} y_l'^2 + \right. \\ \left. + \left(\frac{2}{5-p}\right)^2 (x_i^2 + y_l^2)\right] + \mathcal{O}(L^{-1}), \tag{2.8}$$

where $8(= [p{+}1] + [7{-}p])$ fields x_i $(i = 1, \ldots, p+1)$ and y_l $(l = 1, \ldots, 7-p)$ are dynamical fields remaining after gauge fixing among 10 original bosonic fields within $\Xi^{(1)}$. The σ-dependence can be Fourier transformed by taking

$$X(\tau,\sigma) = \frac{1}{\sqrt{\tilde{\alpha}}} \sum_{n\ge 0} \left[\cos\left(\frac{n}{\tilde{\alpha}}\sigma\right) X_n(\tau) + \sin\left(\frac{n}{\tilde{\alpha}}\sigma\right) X_{-n}(\tau)\right], \tag{2.9}$$

where X denotes x_i or y_l. The equation of motion for each mode is

$$\ddot{X}_n - \left[\bar{r}(\tau)^{p-3}\left(\frac{n}{\tilde{\alpha}}\right)^2 + m^2\right] X_n = 0, \tag{2.10}$$

where

$$\bar{r}(\tau) = \left(\frac{\cosh\tau}{\ell}\right)^{2/(5-p)}, \tag{2.11}$$

$m = 1$ for x_i and $m = \frac{2}{5-p}$ for y_l. We sometimes denote

$$M_n(\tau)^2 \equiv \bar{r}(\tau)^{p-3}\left(\frac{n}{\tilde{\alpha}}\right)^2 + m^2 .$$

Note that $M_n(\tau)^2 = M_n(-\tau)^2$. Hamiltonian for each mode after some rescaling of field is given as

$$H_n^X = \frac{1}{2}\left(P_n^2 + M_n(\tau)^2 X_n^2\right), \tag{2.12}$$

[1] By $z \sim 0$, we mean $z = 1/\Lambda$ with $\Lambda\tilde{\ell} \to \infty$.

where $P_n(= -i\partial\mathcal{L}/\partial\dot{X}_n) = -i\dot{X}_n$. Thus the total hamiltonian for bosonic fluctuations is given by summing up all modes from x_i and y_l as

$$H_{\rm b} = \sum_{I=(i,l)} \sum_{n=-\infty}^{\infty} H_n^I \,. \tag{2.13}$$

For fermionic fluctuations, the quadratic action around the trajectory is also obtained by expanding the GS action [16]. The result for IIA or IIB is given as [2]

$$\begin{aligned} S_{{\rm f},A}^{(2)} &= \frac{1}{2\pi}\int d\tau \int_0^{2\pi\tilde{\alpha}} d\sigma\, \Theta^T\Gamma_0\Big[\sqrt{2}\Gamma_+\partial_\tau\Theta - im_{f(p)}\Gamma_+\Gamma_{(p)}\Gamma_+\Theta - \\ &\qquad\qquad - i\sqrt{2}\,\bar{r}(\tau)^{-\frac{3-p}{2}}\Gamma_{11}\Gamma_+\partial_\sigma\Theta\Big]\,, \qquad (2.14)\\ S_{{\rm f},B}^{(2)} &= \frac{1}{2\pi}\int d\tau \int_0^{2\pi\tilde{\alpha}} d\sigma\, (\Theta^I)^T\Gamma_0\Big[\sqrt{2}\Gamma_+\partial_\tau\Theta^I - im_{f(p)}\Gamma_+\Gamma_{(p)}^{IJ}\Gamma_+\Theta^J + \\ &\qquad\qquad + i\sqrt{2}s\,\bar{r}(\tau)^{-\frac{3-p}{2}}s_2^{IJ}\Gamma_+\partial_\sigma\Theta^J\Big]\,, \qquad (2.15) \end{aligned}$$

where Θ for IIA is 32-component Majorana spinor and Θ^I $(I = 1,2)$ for IIB are two Majorana-Weyl spinors. Matrices s_k^{IJ} $(k = 0,1,2)$ are given by Pauli matrices as $s_0 = -i\sigma_2$, $s_1 = \sigma_1$ and $s_2 = \sigma_3$. We choose the representation of Gamma matrices as

$$\begin{aligned} &\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_{\hat{x}_i} = \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}, \\ &\Gamma_{\hat{y}_l} = \begin{pmatrix} \gamma_{p+1+l} & 0 \\ 0 & -\gamma_{p+1+l} \end{pmatrix}, \qquad \Gamma_{11} = \begin{pmatrix} \gamma_9 & 0 \\ 0 & -\gamma_9 \end{pmatrix} \end{aligned} \tag{2.16}$$

and decompose Θ as

$$\Theta = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{\theta} \\ \theta \end{pmatrix}.$$

We see that only θ components appear in the action (2.14) or (2.15). If we further decompose θ as

$$\theta = \theta_+ + \theta_-\,, \qquad \gamma_9\theta_\pm = \pm\theta_\pm \qquad \text{for IIA}\,, \tag{2.17}$$

$$(\theta^1,\theta^2) = (\theta_-,\theta_+) \qquad \text{or} \qquad s_2\theta_\pm = \mp\theta_\pm \qquad \text{for IIB}\,, \tag{2.18}$$

the action for IIA and IIB is represented in a unified form

$$\begin{aligned} S_{\rm f}^{(2)} &= -\frac{1}{2\pi}\int d\tau \int_0^{2\pi\tilde{\alpha}} d\sigma\, \Big[\theta_+^T\partial_\tau\theta_+ + \theta_-^T\partial_\tau\theta_- - 2im_{f(p)}\theta_+^T\gamma_{(p)}\theta_- - \\ &\qquad\qquad - i\bar{r}(\tau)^{-\frac{3-p}{2}}(\theta_+^T\partial_\sigma\theta_+ - \theta_-^T\partial_\sigma\theta_-)\Big]\,. \end{aligned} \tag{2.19}$$

Here

$$m_{f(p)} = \frac{7-p}{2(5-p)} \tag{2.20}$$

and

$$\gamma_{(p=0)} = \gamma_9\gamma_1\,, \qquad \gamma_{(p=2)} = \gamma_{123}\,, \qquad \gamma_{(p=4)} = -\gamma_9\gamma_{12345}\,, \tag{2.21}$$

$$\gamma_{(p=1)} = s_1\gamma_{12}\,, \qquad \gamma_{(p=3)} = -s_0\gamma_{1234}\,. \tag{2.22}$$

The equation of motion for each Fourier mode $\theta_{\pm,n}(\tau)$ given by

$$\theta_\pm(\tau,\sigma) = \frac{1}{\sqrt{\tilde{\alpha}}}\sum_{n=-\infty}^{\infty}\theta_{\pm,n}(\tau)e^{i\frac{n}{\tilde{\alpha}}\sigma} \tag{2.23}$$

are

$$\partial_\tau\theta_{\pm,n} \pm \frac{n}{\tilde{\alpha}}\bar{r}(\tau)^{-\frac{3-p}{2}}\theta_{\pm,n} - im_{f(p)}\gamma_{(p)}\theta_{\mp,n} = 0\,. \tag{2.24}$$

Hamiltonian is given as

$$H_{\rm f} = \sum_{n=-\infty}^{\infty}\left[-2im_{f(p)}\theta^T_{+,-n}\gamma_{(p)}\theta_{-,n} + \frac{n}{\tilde{\alpha}}\bar{r}(\tau)^{-\frac{3-p}{2}}(\theta^T_{+,-n}\theta_{+,n} - \theta^T_{-,-n}\theta_{-,n})\right]. \tag{2.25}$$

Thus we have found complete hamiltonian $H = H_{\rm b} + H_{\rm f}$ for all the quadratic fluctuations around the classical trajectory.

Finally, note that the final form of the action $S = S^{(2)}_{\rm b} + S^{(2)}_{\rm f}$ coincides with the one obtained for the fluctuations around the real null geodesic in Dp-brane geometry without double Wick rotation.

3. Quantization and the diagonalization of S-matrix

3.1 S-matrix from $\tau = -T$ to $\tau = T$: definition

Now we quantize the system $H = H_{\rm b} + H_{\rm f}$ and analyze the boundary-to-boundary S-matrix along the tunneling null geodesic, which is interpreted as two-point functions of the boundary gauge theory.

For bosonic sector, we write general solutions of (2.10) as

$$X_n(\tau) = f^{(+)}_n(\tau)a_n + f^{(-)}_n(\tau)a^\dagger_n \tag{3.1}$$

with constant operators a_n and $a^\dagger_n$. We choose $f^{(\pm)}_n$ to satisfy the time reflection symmetry [1] $f^{(+)}_n(\tau) = f^{(-)}_n(-\tau)$ which is alternative to the reality condition in the real-time formulation. We also set the 'boundary condition' at $\tau\to\infty$ as $f^{(+)}_n \to 0$ (or at least $f^{(+)}_n/f^{(-)}_n \to 0$) with divergent $f^{(-)}_n$. Furthermore, we impose the normalization condition

$$f^{(+)}_n\frac{df^{(-)}_n}{d\tau} - f^{(-)}_n\frac{df^{(+)}_n}{d\tau} = 1\,. \tag{3.2}$$

Then, the canonical commutation relation $[X_n, P_{n'}] = i\delta_{n,n'}$ becomes equivalent to $[a_n, a^\dagger_{n'}] = \delta_{n,n'}$. The hamiltonian is written by $(a_n, a^\dagger_n)$ as

$$H_{\rm b} = \sum_{I=(i,l)} \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2}\left[-\left(\dot{f}^{I,(+)}_n\right)^2 + (M^I_n)^2 \left(f^{I,(+)}_n\right)^2 \right] (a^I_n)^2 + \right.$$
$$+ \frac{1}{2}\left[-\dot{f}^{I,(+)}_n \dot{f}^{I,(-)}_n + (M^I_n)^2 f^{I,(+)}_n f^{I,(-)}_n \right] \left(a^{I\dagger}_n a^I_n + \frac{1}{2} \right) +$$
$$\left. + \frac{1}{2}\left[-\left(\dot{f}^{I,(-)}_n\right)^2 + (M^I_n)^2 \left(f^{I,(-)}_n\right)^2 \right] (a^{I\dagger}_n)^2 \right\}. \tag{3.3}$$

For fermionic sector, general solutions for (2.24) are expressed by spinor operators d^α_n and $d^{\alpha\dagger}_n$ $(\alpha = 1, \cdots, 8)$ as

$$\left(\theta_{-,n}(\tau), \tilde{\theta}_{+,n}(\tau)\right) = \left(\phi^{(+)}_n(\tau)\, d_n + \phi^{(-)}_n(\tau)\, d^\dagger_n,\ \psi^{(+)}_n(\tau)\, d_n + \psi^{(-)}_n(\tau)\, d^\dagger_n\right) \tag{3.4}$$

where $\tilde{\theta}_{+,n} = i\gamma_{(p)}\theta_{+,n}$. We choose

$$\phi^{(+)}_n \psi^{(-)}_n - \phi^{(-)}_n \psi^{(+)}_n = \frac{1}{2} \tag{3.5}$$

and

$$\phi^{(+)}_n = -\psi^{(+)}_{-n}\,, \qquad \phi^{(-)}_n = \psi^{(-)}_{-n}\,. \tag{3.6}$$

Also, we can set $\phi^{(+)}_{-n}(-\tau) = \phi^{(-)}_n(\tau)$ from the time reflection symmetry $\theta_{\pm,n}(\tau)^\dagger = \theta_{\pm,-n}(-\tau)$ [2]. Furthermore, we set the boundary condition at $\tau \to \infty$ as $\phi^{(+)}_n \to 0$ or $\phi^{(+)}_n/\phi^{(-)}_n \to 0$ with $|\phi^{(-)}_n| \to \infty$. Then the canonical anti-commutation relations for $\theta_{\pm,n}$

$$\{\theta^\alpha_{s,n}, \theta^{\alpha'}_{s',n'}\} = \frac{1}{2}\delta_{s,s'}\delta_{n,-n'}\delta_{\alpha,\alpha'} \tag{3.7}$$

lead to

$$\{d^\alpha_n, d^{\alpha'\dagger}_{n'}\} = \delta_{n,-n'}\delta_{\alpha,\alpha'}\,. \tag{3.8}$$

The hamiltonian is written as

$$H_{\rm f} = \sum_{n=-\infty}^{\infty} \sum_{\alpha=1}^{8} \left\{ -2\left[m_{f(p)} \left(\phi^{(+)}_n\right)^2 + \frac{n}{\tilde{\alpha}} \bar{r}^{\frac{p-3}{2}} \phi^{(+)}_n \phi^{(+)}_{-n} \right] d^\alpha_{-n} d^\alpha_n + \right.$$
$$+ 2\left[m_{f(p)} \left(\phi^{(+)}_n \phi^{(-)}_n + \phi^{(+)}_{-n} \phi^{(-)}_{-n} \right) + \frac{n}{\tilde{\alpha}} \bar{r}^{\frac{p-3}{2}} \left(\phi^{(+)}_{-n} \phi^{(-)}_n - \phi^{(+)}_n \phi^{(-)}_{-n} \right) \right] \times$$
$$\left. \times \left(d^{\alpha\dagger}_{-n} d^\alpha_n - \frac{1}{2} \right) + 2\left[m_{f(p)} \left(\phi^{(-)}_n\right)^2 - \frac{n}{\tilde{\alpha}} \bar{r}^{\frac{p-3}{2}} \phi^{(-)}_n \phi^{(-)}_{-n} \right] d^{\alpha\dagger}_{-n} d^{\alpha\dagger}_n \right\}. \tag{3.9}$$

We define the (euclidean) S-matrix from $\tau = -T$ to $\tau = T$ as the integration of the anti-time ordered product

$$S(T) = \mathcal{T}_- \exp\left[-\int_{-T}^{T} d\tau H(\tau) \right] \tag{3.10}$$

or

$$\frac{dS(T)}{dT} = -H(-T)S(T) - S(T)H(T)\,. \tag{3.11}$$

Note that $S(T)$ is hermitean $S^\dagger(T) = S(T)$ since $H^\dagger(\tau) = H(-\tau)$. (We assume $f_n^{(\pm)}$ and $\phi_n^{(\pm)}$ are real.) We will interpret this S-matrix as the diagonalized two-point functions of BMN-type operators $\mathcal{O}$ of the boundary gauge theory. In general, this hamiltonian $H = H_{\rm b} + H_{\rm f}$ is time-dependent $H = H(\tau)$ and $S(T)$ must be 'diagonalized' by performing time-dependent Bogoliubov transformation in order to extract the information of the diagonalized value of two-point functions for the dual gauge theory.

In the following, we review the diagonalization procedure for bosonic part of the S-matrix developed in ref. [1] and then generalize the discussion to the fermionic part.

3.2 General theory for time-dependent harmonic oscillators

Bosonic part. We consider each mode separately by decomposing the S-matrix as

$$S_{\rm b}(T) = \prod_{I=x_i,y_l} \prod_{n=-\infty}^{\infty} S_{I,n}(T)\,, \tag{3.12}$$

where

$$S_{\rm b}^{I,n}(T) = \prod_{n,I} \mathcal{T}_- \exp\left[-\int_{-T}^{T} d\tau\, H_{\rm b}^{n,I}(\tau)\right]. \tag{3.13}$$

As was discussed in ref. [1], we can represent $S_{\rm b}(T)$ (or $S_{I,n}$) in two ways. One is normal ordered form which is naturally obtained from the definition of S-matrix (3.10) or (3.11) as

$$S_{I,n}(T) = N_{I,n}(T) : \exp\left[\frac{1}{2}A_{I,n}(T)(a^\dagger_{I,n})^2 + B_{I,n}(T)a^\dagger_{I,n}a_{I,n} + \frac{1}{2}C_{I,n}(T)a^2_{I,n}\right] : \tag{3.14}$$

with

$$N^2_{I,n} = 1 + B_{I,n} = \frac{1}{2f^{(-)}_{I,n}\dot{f}^{(-)}_{I,n}}, \qquad A_{I,n} = C_{I,n} = -\frac{1}{2}\left(\frac{f^{(+)}_{I,n}}{f^{(-)}_{I,n}} + \frac{\dot{f}^{(+)}_{I,n}}{\dot{f}^{(-)}_{I,n}}\right). \tag{3.15}$$

Another is exponential form

$$S_{I,n}(T) = \tilde{N}_{I,n}(T) \exp\left[\frac{1}{2}\tilde{A}_{I,n}(T)(a^\dagger_{I,n})^2 + \tilde{B}_{I,n}(T)a^\dagger_{I,n}a_{I,n} + \frac{1}{2}\tilde{A}_{I,n}(T)a^2_{I,n}\right], \tag{3.16}$$

where $\tilde{A}_{I,n}$, $\tilde{B}_{I,n}$ and $\tilde{N}_{I,n}$ are determined by

$$\exp\begin{pmatrix} -\tilde{B}_{I,n} & -\tilde{A}_{I,n} \\ \tilde{A}_{I,n} & \tilde{B}_{I,n} \end{pmatrix} = \frac{1}{1+B_{I,n}}\begin{pmatrix} 1 & -A_{I,n} \\ A_{I,n} & (1+B_{I,n})^2 - A^2_{I,n} \end{pmatrix} \tag{3.17}$$

and[2]

$$\tilde{N}_{I,n} = N_{I,n} \exp\left(\frac{\tilde{B}_{I,n}}{2}\right)(1+B_{I,n})^{-1/2}. \tag{3.18}$$

[2] We correct a minor mistake of eq.(3.42) in [1].

Once $S(T)$ is represented by exponential form, it can be transformed to a 'diagonalized' form by T-dependent Bogoliubov transformation

$$\begin{pmatrix} a^\dagger_{I,n} \\ a_{I,n} \end{pmatrix} \to \begin{pmatrix} b^\dagger_{I,n}(T) \\ b_{I,n}(T) \end{pmatrix} = \begin{pmatrix} G_{I,n}(T) & F_{I,n}(T) \\ E_{I,n}(T) & D_{I,n}(T) \end{pmatrix} \begin{pmatrix} a^\dagger_{I,n} \\ a_{I,n} \end{pmatrix}. \tag{3.19}$$

We choose D, E, F and G ($DG - EF = 1$) in order to satisfy

$$\begin{aligned} S_{I,n}(T) &= \tilde{N}_{I,n} \exp\left(-\frac{\tilde{B}_{I,n}}{2}\right) \exp\left[\frac{1}{2}\left(a^\dagger_{I,n}\ a_{I,n}\right)\begin{pmatrix} \tilde{A}_{I,n} & \tilde{B}_{I,n} \\ \tilde{B}_{I,n} & \tilde{A}_{I,n} \end{pmatrix}\begin{pmatrix} a^\dagger_{I,n} \\ a_{I,n} \end{pmatrix}\right] \\ &= \tilde{N}_{I,n} \exp\left(-\frac{\tilde{B}_{I,n}}{2}\right) \exp\left[-\frac{1}{2}\left(b^\dagger_{I,n}\ b_{I,n}\right)\begin{pmatrix} 0 & \Omega_{I,n} \\ \Omega_{I,n} & 0 \end{pmatrix}\begin{pmatrix} b^\dagger_{I,n} \\ b_{I,n} \end{pmatrix}\right], \end{aligned} \tag{3.20}$$

where

$$\Omega_{I,n} = \sqrt{\tilde{B}^2_{I,n} - \tilde{A}^2_{I,n}} \tag{3.21}$$

or

$$\cosh \Omega_{I,n} = \frac{1}{2}\left(1 + B_{I,n} + \frac{1 - A^2_{I,n}}{1 + B_{I,n}}\right). \tag{3.22}$$

The final form of diagonal S-matrix is

$$S_{\rm b}(T) = \prod_{I,n} \exp\left[-\Omega_{I,n}\left(b^\dagger_{I,n}(T)\, b_{I,n}(T) + \frac{1}{2}\right)\right], \tag{3.23}$$

where

$$[b_{I,n}(T), b^\dagger_{I',n'}(T)] = \delta_{I,I'}\delta_{n,n'}\,. \tag{3.24}$$

Since we have chosen $f^{(+)}_{I,n}/f^{(-)}_{I,n} \to 0$ and $|f^{(-)}_{I,n}| \to \infty$ in the $T \to \infty$ limit, $A \sim 1+B \to 0$ and $(b^\dagger(T), b(T)) \to (a^\dagger, a)$. Thus,

$$S_{\rm b}(T) \stackrel{T\to\infty}{\longrightarrow} \prod_{I,n}\left(2 f^{(-)}_{I,n}(T)\dot{f}^{(-)}_{I,n}(T)\right)^{-\left(a^\dagger_{I,n}a_{I,n}+\frac{1}{2}\right)}. \tag{3.25}$$

Fermionic part. Now we discuss the fermionic part of S-matrix

$$S_{\rm f}(T) = \prod_{\alpha=1}^{8}\prod_{n=-\infty}^{\infty} \mathcal{T}_- \exp\left[-\int_{-T}^{T} d\tau H^n_{\rm f}\right] \left(\equiv \prod_{\alpha,n} S_n(T)\right). \tag{3.26}$$

As in the case of bosonic part, the first step is to write S_n in the normal-ordered form. We can write

$$S_n(T) = N^{\rm f}_n(T) : \exp\left[\frac{1}{2}A^{\rm f}_n(T) d^{\alpha\dagger}_n d^{\alpha\dagger}_{-n} + B^{\rm f}_n(T) d^{\alpha\dagger}_n d^{\alpha}_{-n} + \frac{1}{2}C^{\rm f}_n(T) d^{\alpha}_n d^{\alpha}_{-n}\right] : \tag{3.27}$$

with

$$A^{\rm f}_n = -A^{\rm f}_{-n}\,, \qquad B^{\rm f}_n = B^{\rm f}_{-n}\,, \qquad C^{\rm f}_n = -C^{\rm f}_{-n}\,, \qquad N^{\rm f}_n = N^{\rm f}_{-n}\,. \tag{3.28}$$

Note that S_n and S_{-n} do not commute with each other and we have to treat them pairwise. From the definition, S-matrix satisfies the relation (3.11) and

$$\theta_{-,n}(T) = S(T)^{-1}\theta_{-,n}(-T)S(T)\,, \qquad \tilde{\theta}_{+,n}(T) = S(T)^{-1}\tilde{\theta}_{+,n}(-T)S(T)\,. \tag{3.29}$$

These relations determine $A^{\rm f}_n$, $B^{\rm f}_n$, $C^{\rm f}_n$ and $N^{\rm f}_n$ as

$$1 + B^{\rm f}_n = \frac{1}{(N^{\rm f}_n)^2} = \frac{1}{2}\frac{1}{\left(\phi^{(-)}_n\right)^2 + \left(\phi^{(-)}_{-n}\right)^2}\,, \qquad A^{\rm f}_n = C^{\rm f}_n = \frac{\phi^{(+)}_{-n}\phi^{(-)}_{-n} - \phi^{(+)}_n\phi^{(-)}_n}{\left(\phi^{(-)}_n\right)^2 + \left(\phi^{(-)}_{-n}\right)^2}\,. \tag{3.30}$$

Second step is to represent $S_n(T)$ in the exponential form

$$S_{\rm f}(T) = \left(\prod_{\alpha,n}\tilde{N}^{\rm f}_n(T)\right)\exp\left\{\sum_{\alpha,n}\left[\frac{1}{2}\tilde{A}^{\rm f}_n(T)d^{\alpha\dagger}_n d^{\alpha\dagger}_{-n} + \tilde{B}^{\rm f}_n(T)d^{\alpha\dagger}_n d^{\alpha}_{-n} + \frac{1}{2}\tilde{A}^{\rm f}_n(T)d^{\alpha}_n d^{\alpha}_{-n}\right]\right\}\,, \tag{3.31}$$

where $\tilde{A}^{\rm f}_n$, $\tilde{B}^{\rm f}_n$ and $\tilde{N}^{\rm f}_n$ also satisfy the relation corresponding to (3.28). The relation between $(A^{\rm f}_n, B^{\rm f}_n, N^{\rm f}_n)$ and $(\tilde{A}^{\rm f}_n, \tilde{B}^{\rm f}_n, \tilde{N}^{\rm f}_n)$ is determined explicitly in the appendix and the result is

$$\begin{aligned}
\tilde{N}^{\rm f}_n &= N^{\rm f}_n \exp\left(-\frac{\tilde{B}^{\rm f}_n}{2}\right)\sqrt{1 + B^{\rm f}_n} && (3.32)\\
\cosh\Omega^{\rm f}_n &= \frac{1}{2}\left(1 + B^{\rm f}_n + \frac{1 + (A^{\rm f}_n)^2}{1 + B^{\rm f}_n}\right)\,, && (3.33)\\
\tilde{B}^{\rm f}_n\frac{\sinh\Omega^{\rm f}_n}{\Omega^{\rm f}_n} &= \frac{1}{2}\left(1 + B^{\rm f}_n - \frac{1 - (A^{\rm f}_n)^2}{1 + B^{\rm f}_n}\right)\,, && (3.34)\\
\tilde{A}^{\rm f}_n\frac{\sinh\Omega^{\rm f}_n}{\Omega^{\rm f}_n} &= \frac{A^{\rm f}_n}{1 + B^{\rm f}_n} && (3.35)
\end{aligned}$$

with

$$\Omega^{\rm f}_n = \sqrt{(\tilde{A}^{\rm f}_n)^2 + (\tilde{B}^{\rm f}_n)^2}\,. \tag{3.36}$$

Note that the above equations have a symmetry for $\Omega^{\rm f}_n \to -\Omega^{\rm f}_n$: We fix the ambiguity by imposing the condition $\Omega^{\rm f}_n(T \to \infty) \to \infty$. Finally, we can convert this to the diagonalized form

$$S_{\rm f}(T) = \prod_{\alpha,n}\exp\left[-\Omega^{\rm f}_n(T)\left(c^{\alpha\dagger}_n(T)\,c^{\alpha}_{-n}(T) - \frac{1}{2}\right)\right] \tag{3.37}$$

with T-dependent operators c^{α}_n and $c^{\alpha\dagger}_n$ satisfying

$$\{c^{\alpha}_n(T), c^{\alpha'\dagger}_{n'}(T)\} = \delta_{\alpha,\alpha'}\delta_{n,-n'}\,. \tag{3.38}$$

For $T \to \infty$, $\exp(\Omega^{\rm f}_n) \to 1/(1+B^{\rm f}_n)$ since we have chosen $\phi^{(+)}_n/\phi^{(-)}_n \to 0$ and $|\phi^{(-)}_n| \to \infty$ in this limit. Thus,

$$S_{\rm f}(T) \overset{T\to\infty}{\longrightarrow} \prod_{\alpha,n}\left(2\left[(\phi^{(-)}_n)^2 + (\phi^{(-)}_{-n})^2\right]\right)^{-\left(d^{\alpha\dagger}_n d^{\alpha}_{-n} - \frac{1}{2}\right)}\,. \tag{3.39}$$

4. S-matrices as two-point functions

To summarize the result of the previous section, S-matrix $S(T) = S_{\rm b}(T)S_{\rm f}(T)$ from $\tau = -T$ to $\tau = T$ is given by

$$\begin{aligned} S(T) &= \prod_{I,n} \exp\left[-\Omega_{I,n}\left(b^\dagger_{I,n}(T)b_{I,n}(T)+\frac{1}{2}\right)\right]\prod_{\alpha,n}\exp\left[-\Omega^{\rm f}_n(T)\left(c^{\alpha\dagger}_n(T)\,c^\alpha_{-n}(T)-\frac{1}{2}\right)\right] \\ &\xrightarrow{T\to\infty} \prod_{I,n}\left(2f^{(-)}_{I,n}(T)\dot{f}^{(-)}_{I,n}(T)\right)^{-\left(a^\dagger_{I,n}a_{I,n}+\frac{1}{2}\right)}\prod_{\alpha,n}\left(2\left[(\phi^{(-)}_n)^2+(\phi^{(-)}_{-n})^2\right]\right)^{-\left(d^{\alpha\dagger}_n d^\alpha_{-n}-\frac{1}{2}\right)}. \end{aligned} \quad (4.1)$$

In the previous papers [1, 2], we gave our prescription of obtaining two-point functions $\langle\bar{\mathcal{O}}(t_f)\mathcal{O}(t_i)\rangle$ from the above S-matrix $S(T)$: The two-point function of a certain operator $\mathcal{O}(\vec{x})$ is given by $S(T)$ (written in terms of occupation numbers N^I_n of each modes) multiplying the classical and spin angular momentum contribution. The operator $\mathcal{O}(\vec{x})$ is determined by N^I_n and the bulk quantity T or ℓ is naturally replaced by boundary quantity by using the relations[3] $e^T \sim |t_f - t_i|\Lambda$ and $2\tilde{\ell} \sim |t_f - t_i|$.

For example, in [1, 2], we mainly analyze supergravity $n = 0$ modes and find that

$$\begin{aligned} S^{(n=0)}(T) &= \prod_{i=1}^{p+1} e^{-2\left(a^\dagger_{x_i,n}a_{x_i,n}+\frac{1}{2}\right)T}\prod_{l=1}^{7-p} e^{-\frac{4}{5-p}\left(a^\dagger_{y_l,n}a_{y_l,n}+\frac{1}{2}\right)T}\prod_{\alpha=1}^{8} e^{-\frac{7-p}{5-p}\left(d^{\alpha\dagger}_0 d^\alpha_0-\frac{1}{2}\right)T} \\ &= \exp\left[\left(-2N^{b,x}_0 - \frac{4}{5-p}N^{b,y}_0 - \frac{7-p}{5-p}N^f_0 - \frac{(3-p)^2}{5-p}\right)T\right] \end{aligned} \quad (4.2)$$

for $T \to \infty$. Here N^x_0 and N^y_0 are the total occupation numbers of $n = 0$ modes for x_i and y_l oscillators respectively. Also, N^f_0 is the occupation number of fermionic oscillators. Then, by including the classical and spin angular momentum contribution $\exp[-\frac{4}{5-p}(J+\frac{1}{2}N^f_0)T]$, we identify $S^{(n=0)}(T)$ with the two-point functions for $\mathcal{O}(\vec{x})$ of the boundary gauge theory as

$$\langle\bar{\mathcal{O}}(t_f,0)\mathcal{O}(t_i,0)\rangle \sim (|t_f-t_i|\Lambda)^{-\frac{4}{5-p}(J+\frac{1}{2}N^f_0)-2N^{b,x}_0-\frac{4}{5-p}N^{b,y}_0-\frac{7-p}{5-p}N^f_0-\frac{(3-p)^2}{5-p}}. \quad (4.3)$$

Here $\mathcal{O}(\vec{x})$ is the BMN-type operator [3] consisting of large number J of $Z(=\phi_{8-p}+i\phi_{9-p})$ with N^{b,x_i}_0, N^{b,y_l}_0 and $N^{f,\alpha}_0$ numbers of D_i, ϕ_l's and χ^α's respectively: They are arranged in a trace symmetrizely. Here ϕ_i ($i = 1, \cdots, 9-p$) represent U(N) adjoint scalars and χ^α the half of the 16 spinors of the $(p+1)$-dimensional gauge theory. The result precisely agrees with the analysis from supergravity theory [10, 11, 13, 14] where the correlation functions are determined via holographic relation (GKP/Witten relation) given in [5, 6].

In the rest of this section, we analyze the $n \neq 0$ modes using the technique developed in the previous section.

---

[3]We set cutoff at $z(\tau = \pm T) = 1/\Lambda$, i.e., $\cosh T = \tilde{\ell}\Lambda$. Thus for $T \to \infty$, $e^T \sim 2\tilde{\ell}\Lambda$. Recall that $|t_f - t_i| \sim 2\tilde{\ell}$, we see $e^T \sim |t_f - t_i|\Lambda$. We also note that we have to keep $\Lambda \sim 1$ since we are in the near-horizon region $z \geq 1$ of the background.

4.1 Perturbative analysis for $n \neq 0$ modes

To obtain S-matrix (4.1) explicitly, we need general solutions for the equations of motion (2.10) and (2.24). For $n \neq 0$ and $p \neq 3$, it is difficult to obtain exact solutions.[4] To obtain asymptotic solutions for $\tau \to \infty$ is easier. However, it is not enough by itself since we need solutions satisfying the time reflection symmetry. Here we instead use the perturbative method by using $|n|/\tilde{\alpha}$ $(= \frac{2}{5-p}\frac{|n|L^2}{J})$ as an expansion parameter.

Bosonic case. By expanding the field $X_n(\tau)$ with $n \neq 0$ as

$$X_n = X_n^{(0)} + \left(\frac{n}{\tilde{\alpha}}\right)^2 X_n^{(2)} + \cdots + \left(\frac{n}{\tilde{\alpha}}\right)^{2i} X_n^{(2i)} + \cdots\,, \tag{4.4}$$

equations of motion (2.10) at the order $(n/\tilde{\alpha})^{2i}$ for $i \geq 0$ is written as

$$\ddot{X}_n^{(2i)} - \bar{r}(\tau)^{p-3} X_n^{(2i-2)} - m^2 X_n^{(2i)} = 0\,, \tag{4.5}$$

where we assign $X_n^{(-2)} = 0$. These are solved recursively as

$$X_n^{(2i)} = \frac{1}{2m}\left[e^{m\tau}\left(\int e^{-m\tau}\bar{r}^{p-3}X_n^{(2i-2)}d\tau\right) - e^{-m\tau}\left(\int e^{m\tau}\bar{r}^{p-3}X_n^{(2i-2)}d\tau\right)\right]. \tag{4.6}$$

In terms of $f_n^{(\pm:2i)}$ with time reflection symmetry $f_n^{(+:2i)}(\tau) = f_n^{(-:2i)}(-\tau)$, solutions are written as

$$\begin{aligned} f_n^{(\pm:2i)}(\tau) = \frac{1}{2m}\Bigg[&e^{m\tau}\left(\int_0^\tau e^{-m\tau}\bar{r}^{p-3}f_n^{(\pm:2i-2)}d\tau + c_n^{(\pm,2i)}\right) + \\ &+ e^{-m\tau}\left(-\int_0^\tau e^{m\tau}\bar{r}^{p-3}f_n^{(\pm:2i-2)}d\tau + c_n^{(\mp,2i)}\right)\Bigg], \end{aligned} \tag{4.7}$$

where $c_n^{(\pm,2i)}$ are constants. These constants can be usually determined by the normalization condition (3.2) and the boundary condition. At the 0-th order, $c_n^{(+,0)} = 0$ from the boundary condition at large T. Then the normalization condition up to the order $(n/\tilde{\alpha})^2$ is

$$\begin{aligned} 1 &= f_n^{(+)}\dot{f}_n^{(-)} - f_n^{(-)}\dot{f}_n^{(+)} \\ &= \frac{1}{2m}(c_n^{(-,0)})^2 + \frac{1}{m}c_n^{(-,0)}c_n^{(-,2)}\left(\frac{n}{\tilde{\alpha}}\right)^2 + \mathcal{O}\left(\left(\frac{n}{\tilde{\alpha}}\right)^4\right). \end{aligned} \tag{4.8}$$

Thus, we can choose

$$c_n^{(-,0)} = \sqrt{2m}\,, \qquad c_n^{(-,2)} = 0\,. \tag{4.9}$$

For $p < 4$, the boundary condition $f_n^{(+:2)}(\tau \to \infty) \to 0$ is satisfied if we choose $c_n^{(+,2)}$ as

$$c_n^{(+,2)} = \frac{1}{\sqrt{2m}}\int_\infty^0 e^{-2m\tau}\bar{r}^{p-3}d\tau\,. \tag{4.10}$$

For $p = 4$, $f_n^{(+:2)}/f_n^{(-:2)}|_{\tau\to\infty} \to 0$ is satisfied for any value of $c_n^{(+,2)}$, though $f_n^{(+:2)}(\tau \to \infty)$ itself diverges. In fact, the value of $c_n^{(+,2)}$ does not affect the final result of S-matrix at large T.

[4] For $p = 4$, the equation of motion for bosonic fluctuation (2.10) can be solved by Mathieu functions.

To summarize, $f_n^{(\pm)}$ up to the order $(n/\tilde{\alpha})^2$ for $p<4$ is

$$f_{n(p<4)}^{(\pm)} = \frac{1}{\sqrt{2m}} e^{\mp m\tau} \pm \frac{1}{(2m)^{3/2}} \left[e^{\pm m\tau} \int_{\pm\infty}^{\tau} e^{\mp 2m\tau} \bar{r}^{p-3} d\tau - e^{\mp m\tau} \int_0^{\tau} \bar{r}^{p-3} d\tau \right] \left(\frac{n}{\tilde{\alpha}}\right)^2 + \cdots \tag{4.11}$$

with $m=1$ for x_i and $m=\frac{2}{5-p}$ for y_l. For $p<3$,

$$2f_n^{(-)}(T)\dot{f}_n^{(-)}(T) \stackrel{T\to\infty}{\longrightarrow} e^{2mT} \left[1 + \frac{1}{4m} (2\ell)^{\frac{2(3-p)}{5-p}} B\left(\frac{3-p}{5-p}, \frac{3-p}{5-p}\right) \left(\frac{n}{\tilde{\alpha}}\right)^2 \right] + \cdots \tag{4.12}$$

and the S-matrix up to the order $(n/\tilde{\alpha})^2$ (for each mode X_n) is

$$\begin{aligned} S_n^{p<3}(T) \sim & \Bigg((|t_f - t_i|\Lambda)^{2m} \times \\ & \times \left[1 + \frac{1}{4m} \left(\frac{5-p}{2}\right)^{\frac{2(3-p)}{5-p}} B\left(\frac{3-p}{5-p}, \frac{3-p}{5-p}\right) \left(\frac{n}{\tilde{\alpha}}\right)^2 |t_f - t_i|^{\frac{2(3-p)}{5-p}} \right] \Bigg)^{-\left(a_n^\dagger a_n + \frac{1}{2}\right)} . \end{aligned} \tag{4.13}$$

Here $B(p,q)$ is the Beta function defined by

$$B(p,q) = \int_0^\infty \frac{x^{q-1}}{(1+x)^{p+q}} dx \, .$$

In general, $S_n^{p<3}$ is represented as

$$S_n^{p<3}(T) \sim \left((|t_f - t_i|\Lambda)^{2m} \left[\sum_{i\geq 0} |t_f - t_i|^{\frac{2(3-p)}{5-p} i} \left(\frac{L^4 n^2}{J^2}\right)^i g_i^{(b)} \right] \right)^{-\left(a_n^\dagger a_n + \frac{1}{2}\right)} , \tag{4.14}$$

where $g_i^{(b)}$ are numerical constants with $g_0^{(b)} = 1$ and $g_1^{(b)} = \frac{1}{4m} (\frac{5-p}{2})^{-\frac{4}{5-p}} B(\frac{3-p}{5-p}, \frac{3-p}{5-p})$. This expansion is valid for small

$$|t_f - t_i|^{\frac{2(3-p)}{5-p}} \frac{L^4 n^2}{J^2} \, . \tag{4.15}$$

For comparison, the result for $p=3$ is

$$\begin{aligned} 2f_n^{(-)}(T)\dot{f}_n^{(-)}(T) &= e^{2T} \left(1 + \left(\frac{n}{\tilde{\alpha}}\right)^2 T \right) + \cdots \\ &\sim (|t_f - t_f|\Lambda)^2 \left[1 + \left(\frac{n}{\tilde{\alpha}}\right)^2 \ln(|t_f - t_f|\Lambda) \right] + \cdots \\ &\sim (|t_f - t_i|\Lambda)^{2 + (\frac{n}{\tilde{\alpha}})^2 + \cdots} . \end{aligned} \tag{4.16}$$

This is naturally given by the expansion of the known exact result $(|t_f - t_f|\Lambda)^{2\sqrt{1+(n/\alpha)^2}}$.

For $p=4$,

$$2f_n^{(-)}(T)\dot{f}_n^{(-)}(T) = e^{2mT} \left[1 + \frac{1}{m} \left(\frac{n}{\tilde{\alpha}}\right)^2 \tilde{\ell}^{-2} (2T + \sinh(2T)) \right] + \cdots \tag{4.17}$$

and thus

$$S_n^{p=4}(T) \sim \left((|t_f - t_i|\Lambda)^{2m} \left[1 + \frac{2}{m} \Lambda^2 \left(\frac{L^2 n}{J}\right)^2 + \cdots \right] \right)^{-\left(a_n^\dagger a_n + \frac{1}{2}\right)} . \tag{4.18}$$

This expansion is valid if $L^2|n|/J$ is small since $\Lambda \sim 1$.

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Fermionic case. For fermions, we expand the fields $\theta_{\pm,n}$ as

$$
\begin{aligned}
\tilde{\theta}_{+,n} &= \theta^{(0)}_{+,n} + \frac{n}{\tilde{\alpha}}\,\theta^{(1)}_{+,n} + \cdots + \left(\frac{n}{\tilde{\alpha}}\right)^i \theta^{(i)}_{+,n} + \cdots\,, \\
\theta_{-,n} &= \theta^{(0)}_{-,n} + \frac{n}{\tilde{\alpha}}\,\theta^{(1)}_{-,n} + \cdots + \left(\frac{n}{\tilde{\alpha}}\right)^i \theta^{(i)}_{-,n} + \cdots\,,
\end{aligned} \tag{4.19}
$$

where $\tilde{\theta}_{+,n} = i\gamma_{(p)}\theta_{+,n}$. The equations of motion (2.24) at the order $(n/\tilde{\alpha})^i$ is

$$
\dot{\theta}^{(i)}_{\pm,n} \pm \bar{r}^{\frac{p-3}{2}}\theta^{(i-1)}_{\pm,n} - m_{f(p)}\theta^{(i)}_{\mp,n} = 0 \tag{4.20}
$$

with $\theta^{(i<0)}_{\pm,n} = 0$. As for the bosonic case, these are solved recursively. In terms of $\phi^{(+:i)}_n$ and $\psi^{(+:i)}_n$ of (3.4), the solution can be represented as

$$
\begin{aligned}
\phi^{(+:i)}_n &= \frac{1}{2}\,e^{m_{f(p)}\tau}\int e^{-m_{f(p)}\tau}\bar{r}^{\frac{p-3}{2}}\left(\phi^{(+:i-1)}_n - \psi^{(+:i-1)}_n\right)d\tau\, + \\
&\quad + \frac{1}{2}\,e^{-m_{f(p)}\tau}\int e^{m_{f(p)}\tau}\bar{r}^{\frac{p-3}{2}}\left(\phi^{(+:i-1)}_n + \psi^{(+:i-1)}_n\right)d\tau\,,
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
\psi^{(+:i)}_n &= \frac{1}{m_{f(p)}}\left(\dot{\phi}^{(+:i)}_n - \bar{r}^{\frac{p-3}{2}}\phi^{(+:i-1)}_n\right) \\
&= \frac{1}{2}\,e^{m_{f(p)}\tau}\int e^{-m_{f(p)}\tau}\bar{r}^{\frac{p-3}{2}}\left(\phi^{(+:i-1)}_n - \psi^{(+:i-1)}_n\right)d\tau\, - \\
&\quad - \frac{1}{2}\,e^{-m_{f(p)}\tau}\int e^{m_{f(p)}\tau}\bar{r}^{\frac{p-3}{2}}\left(\phi^{(+:i-1)}_n + \psi^{(+:i-1)}_n\right)d\tau\,.
\end{aligned} \tag{4.22}
$$

If we determine $\phi^{(+:i)}_{n\geq 0}$ and $\psi^{(+:i)}_{n\geq 0}$ from the above equations, the remaining $\phi^{(\pm:i)}_{n<0}$ and $\psi^{(\pm:i)}_{n<0}$ can be obtained by using the conditions (3.6) and the time reflection symmetry:

$$
\phi^{(+:i)}_{-n}(-\tau) = (-)^i\phi^{(-:i)}_n(\tau)\,; \qquad \phi^{(+:i)}_n = (-)^{i-1}\psi^{(+:i)}_{-n}\,, \qquad \phi^{(-:i)}_n = (-)^i\psi^{(-:i)}_{-n}\,. \tag{4.23}
$$

In fact, we can adjust the integration constants in (4.21) and (4.22) to satisfy

$$
\phi^{(\pm:i)}_n = \phi^{(\pm:i)}_{-n}\,. \tag{4.24}
$$

Then the normalization condition (3.5) reduces to

$$
\begin{aligned}
\frac{1}{2} &= \phi^{(+)}_n\psi^{(-)}_n - \phi^{(-)}_n\psi^{(+)}_n \\
&= 2\sum_{i=0}^{\infty}\left(\frac{n}{\tilde{\alpha}}\right)^{2i}\left[\sum_{j=0}^{2i}\phi^{(+:j)}_n(0)\,\phi^{(+:2i-j)}_n(0)\right].
\end{aligned} \tag{4.25}
$$

By taking into account the boundary condition, the solution up to the order $(n/\tilde{\alpha})^2$ is

$$
\phi^{(+)}_{\pm n}(\tau) = \phi^{(+:0)}_{\pm n}(\tau) + \frac{n}{\tilde{\alpha}}\,\theta^{(1)}_{+,n}\phi^{(+:1)}_{\pm n}(\tau) + \left(\frac{n}{\tilde{\alpha}}\right)^2\phi^{(+:2)}_{\pm n}(\tau) + \cdots \tag{4.26}
$$

with

$$\phi^{(+:0)}_{\pm n}(\tau) = \frac{1}{2}\, e^{-m_{f(p)}\tau}\,, \tag{4.27}$$

$$\phi^{(+:1)}_{\pm n}(\tau) = \frac{1}{2}\, e^{m_{f(p)}\tau} \int_{\infty}^{\tau} e^{-2m_{f(p)}\tau}\, \bar{r}^{\frac{p-3}{2}}\, d\tau\,, \tag{4.28}$$

$$\begin{aligned}\phi^{(+:2)}_{\pm n}(\tau) &= \frac{1}{2}\, e^{-m_{f(p)}\tau} \int_{0}^{\tau} e^{2m_{f(p)}\tau}\, \bar{r}^{\frac{p-3}{2}} \left(\int_{\infty}^{\tau} e^{-2m_{f(p)}\tau}\, \bar{r}^{\frac{p-3}{2}} d\tau\right) d\tau\, - \\ &\quad - \frac{1}{4}\, e^{-m_{f(p)}\tau} \left(\int_{0}^{\infty} e^{-2m_{f(p)}\tau}\, \bar{r}^{\frac{p-3}{2}} d\tau\right)^2 .\end{aligned} \tag{4.29}$$

For $p < 3$,

$$\begin{aligned}&2\Big((\phi^{(-)}_{n}(T))^2 + (\phi^{(-)}_{-n}(T))^2\Big) \rightarrow \\ &\overset{T\to\infty}{\longrightarrow} e^{2m_{f(p)}T} \left[1 + (2\tilde{\ell})^{\frac{2(3-p)}{5-p}} \left(\frac{2}{5-p}\right)^{\frac{4}{5-p}} \left(N_p - \frac{1}{4}(2^{\frac{4}{5-p}} - 1)\right) \left(\frac{n}{\tilde{\alpha}}\right)^2\right] + \cdots\,,\end{aligned} \tag{4.30}$$

where N_p is a constant given by

$$N_p = \frac{2}{5-p} \int_1^{\infty} (x^2+1)^{\frac{p-1}{5-p}} x^{\frac{p-9}{5-p}}\,. \tag{4.31}$$

Thus the S-matrix up to the order $(n/\tilde{\alpha})^2$ is

$$\begin{aligned}S^{p<3}_{\mathrm{f};n}(T) \sim \Bigg((|t_f - t_i|\Lambda)^{\frac{7-p}{5-p}} \Bigg[1 + |t_f - t_i|^{\frac{2(3-p)}{5-p}} \left(\frac{2}{5-p}\right)^{\frac{4}{5-p}} \times \\ \times \left(N_p - \frac{1}{4}(2^{\frac{4}{5-p}} - 1)\right) \left(\frac{n}{\tilde{\alpha}}\right)^2\Bigg]\Bigg)^{-\left(d^{\dagger}_n d_{-n} - \frac{1}{2}\right)} .\end{aligned} \tag{4.32}$$

As for the bosonic case, if $|t_f - t_i|^{\frac{2(3-p)}{5-p}} \frac{L^4 n^2}{J^2}$ is small enough, we can in principle represent $S^{p<3}_{\mathrm{f};n}(T)$ as

$$S^{p<3}_{\mathrm{f};n}(T) \sim \left((|t_f - t_i|\Lambda)^{\frac{(7-p)}{5-p}} \left[\sum_{i\geq 0} |t_f - t_i|^{\frac{2(3-p)}{5-p}i} \left(\frac{L^4 n^2}{J^2}\right)^i g^{(f)}_i\right]\right)^{-\left(d^{\dagger}_n d_{-n} + \frac{1}{2}\right)} . \tag{4.33}$$

For $p = 4$,

$$2\left[(\phi^{(-)}_{n}(T))^2 + (\phi^{(-)}_{-n}(T))^2\right] \overset{T\to\infty}{\longrightarrow} e^{3T} \left[1 + \frac{5}{16}\tilde{\ell}^{-2} e^{2T} \left(\frac{n}{\tilde{\alpha}}\right)^2\right] + \cdots \tag{4.34}$$

and we see that if $L^2|n|/J$ is small, the S-matrix for this part becomes

$$S^{p=4}_{\mathrm{f};n}(T) \sim \left((|t_f - t_i|\Lambda)^3 \left[1 + \frac{5}{16}\Lambda^2 \left(\frac{n}{\tilde{\alpha}}\right)^2 + \cdots\right]\right)^{-\left(d^{\dagger}_n d_{-n} - \frac{1}{2}\right)} . \tag{4.35}$$

For $p = 3$, we obtain the expansion of the known exact result

$$S^{p=3}_{\mathrm{f};n}(T) \sim (|t_f - t_i|\Lambda)^{-\sqrt{1+(n/\alpha)^2}(2d^{\dagger}_n d_{-n} - 1)} \tag{4.36}$$

for the same analysis.

4.2 $|n|/\tilde{\alpha} \to \infty$ limit

We will briefly give the analysis of S-matrix at large $(n/\tilde{\alpha})^2$. For this purpose, it is convenient to rewrite the equations of motion by making the redefinition of fields $(x_i, y_l) \to (X_i, Y_l)$ and $(\tau, \sigma) \to (\tau_c, \sigma_c)$ by

$$X_i = \frac{2}{5-p}\bar{r}^{-\frac{3-p}{4}} x_i\,, \qquad Y_l = \bar{r}^{-\frac{3-p}{4}} y_l\,, \tag{4.37}$$

$$\sigma = \frac{5-p}{2}\ell\sigma_c\,, \qquad \frac{d\tau}{d\tau_c} = \frac{5-p}{2}\ell\bar{r}^{\frac{3-p}{2}}\,. \tag{4.38}$$

Note that this definition of the world-sheet fields corresponds to the one with the conformal gauge fixing $\sqrt{h}h^{\alpha\beta} = \delta^{\alpha\beta}$. Then the equations motion for bosonic fields (2.10) become

$$\frac{d^2X_n}{d\tau_c^2} - \left[\left(\frac{n}{\alpha}\right)^2\ell^2 + m_X(\tau_c)^2\right]X_n = 0 \tag{4.39}$$

where $\alpha = J/L^2$ and

$$m_X^2(\tau_c) = -\frac{(7-p)}{16\bar{r}^2}[(3-p) + (3p-13)\ell^2\bar{r}^{5-p}]\,, \tag{4.40}$$

$$m_Y^2(\tau_c) = -\frac{(7-p)}{16\bar{r}^2}[(3-p) - (p+1)\ell^2\bar{r}^{5-p}]\,. \tag{4.41}$$

Also, (2.24) becomes

$$\frac{d}{d\tau_c}\theta_{\pm,n} \pm \ell\frac{n}{\alpha}\theta_{\pm,n} - im_{f(p)}\frac{5-p}{2}\ell\bar{r}^{\frac{3-p}{2}}\gamma_{(p)}\theta_{\mp,n} = 0\,. \tag{4.42}$$

For $(n/\alpha)^2 \gg 1$, these equations of motion reduce to

$$\frac{d^2X_n}{d\tau_c^2} - \left(\ell\frac{n}{\alpha}\right)^2 X_n = 0 \qquad \text{and} \qquad \frac{d}{d\tau_c}\theta_{\pm,n} \pm \ell\frac{n}{\alpha}\theta_{\pm,n} = 0\,, \tag{4.43}$$

which are immediately solved as

$$X_n = C_\pm \exp\left(\pm\frac{n}{\alpha}\ell\tau_c\right) \qquad \text{and} \qquad \theta^\alpha_{\pm,n} = \tilde{C}^\alpha_\pm \exp\left(\mp\frac{n}{\alpha}\ell\tau_c\right). \tag{4.44}$$

Thus, in terms of $f_n^{(\pm)}$ and $\phi_n^{(\pm)}$,

$$f_n^{(\pm)} = \sqrt{\frac{\alpha}{2|n|\ell}}\exp\left(\mp\frac{|n|}{\alpha}\ell\tau_c\right) \qquad \text{and} \qquad (\phi_n^{(+)}, \phi_n^{(-)}) = \left(0, -\frac{1}{\sqrt{2}}\exp\left(\frac{|n|}{\alpha}\ell\tau_c\right)\right). \tag{4.45}$$

Then the S-matrix for bosonic part becomes

$$\begin{aligned} S^{p<3}_{\mathrm{b},n}(T_c) \overset{T_c\to T_c^{(\infty)}\,(T\to\infty)}{\longrightarrow} &\exp\left[2\ell\frac{|n|}{\alpha}T_c^{(\infty)}\left(a_n^\dagger a_n + \frac{1}{2}\right)\right] \\ \sim\quad &\exp\left[-2c^{(0)}\frac{L^2|n|}{J}|t_f - t_i|^{\frac{3-p}{5-p}}\left(a_n^\dagger a_n + \frac{1}{2}\right)\right], \end{aligned} \tag{4.46}$$

where $c^{(0)}$ $(<\infty)$ is a constant given by

$$\begin{aligned} T_c^{(\infty)} &= \int_{\ell^{-2/(5-p)}}^{\infty} \frac{1}{\sqrt{\ell^2 r^{5-p}-1}} dr \\ &= \ell^{-\frac{2}{5-p}} \frac{1}{5-p} \int_1^{\infty} \frac{1}{\sqrt{u-1}} u^{-\frac{4-p}{5-p}} du \\ &\equiv \ell^{\frac{2}{5-p}} c^{(0)}. \end{aligned} \tag{4.47}$$

For $p=4$,

$$\begin{aligned} S_{\mathrm{b},n}^{p=4}(T_c) &\stackrel{T_c\to T_c^{(\infty)}}{\longrightarrow} \exp\left[-2\ell\frac{|n|}{\alpha}T_c^{(\infty)}\left(a_n^\dagger a_n+\frac{1}{2}\right)\right] \\ &\sim \quad \exp\left[-8\frac{L^2|n|}{J}\Lambda\left(a_n^\dagger a_n+\frac{1}{2}\right)\right], \end{aligned} \tag{4.48}$$

where we have used $\ell T_c^\infty \sim 4\Lambda$ at $z=1/\Lambda$ $(T\to\infty)$, which comes from the relation $\bar{r}(\tau_c)=\frac{\ell^2}{4}\tau_c^2+\frac{1}{\ell^2}$ for $p=4$. We see that there is no $|t_f-t_i|$ dependence in $S_{\mathrm{b},n}^{p=4}$.

We can calculate the contribution of $\alpha/|n|$ by using the similar perturbative method given in the previous subsection. The result for $p=4$ is

$$S_{\mathrm{b},n}^{p=4} \sim \exp\left\{\left[-8\frac{L^2|n|}{J}\Lambda - \frac{3}{128}c^{(1)}\frac{J}{L^2|n|}|t_f-t_i|+\cdots\right]\left(a_n^\dagger a_n+\frac{1}{2}\right)\right\}, \tag{4.49}$$

where $c^{(1)}=1$ for x_i and $c^{(1)}=9$ for y_l. On the other hand, for $p<3$, the $\alpha/|n|$ correction gives only a contribution without $|t_f-t_i|$ dependence, which is the same phenomena as the cases of $|n|/\alpha$ expansion for $p=4$ given in (4.34).

Similarly, $S_{\mathrm{f},n}$ for fermionic part becomes

$$S_{\mathrm{f},n}^{p<3}(T_c) \stackrel{T_c\to T_c^{(\infty)}}{\longrightarrow} \exp\left[-2c^{(0)}\frac{L^2|n|}{J}|t_f-t_i|^{\frac{3-p}{5-p}}\left(b_n^\dagger b_{-n}-\frac{1}{2}\right)\right] \tag{4.50}$$

and

$$S_{\mathrm{f},n}^{p=4} \stackrel{T_c\to T_c^{(\infty)}}{\longrightarrow} \exp\left[-8\frac{L^2|n|}{J}\Lambda\right]\left(b_n^\dagger b_{-n}-\frac{1}{2}\right). \tag{4.51}$$

Note that the constant $c^{(0)}$ appearing in (4.50) is the same as for the bosonic part given in (4.47).

We can also compute the $\alpha/|n|$ correction to the above result: The correction terms for each p has the same form as the bosonic case, though the coefficients corresponding to $c^{(1)}$ are different from bosonic and fermionic contributions.

4.3 Interpretation as gauge theory correlators

We briefly discuss the interpretation of the above results as two-point functions between $(t_f, x_f^a)=(t_f,0)$ and $(t_i,x_i^a)=(t_i,0)$ for the dual gauge theory.

First, for $p\neq 3$, we see that there appears a dimensionful quantity $n^2/\alpha^2 = L^4n^2/J^2$ (or $n^2/\tilde{\alpha}^2=(\frac{2}{5-p})^2 n^2L^4/J^2$) in the S-matrix other than $|t_f-t_i|=2\tilde{\ell}$. Thus, we characterize the IR or UV behavior of the dual gauge theory by measuring the dimensionless quantity

$$|t_f-t_i|^{\frac{2(3-p)}{5-p}}\frac{L^4n^2}{J^2}\,. \tag{4.52}$$

For $p < 3$, if $|t_f - t_i|^{\frac{2(3-p)}{5-p}} \frac{L^4 n^2}{J^2}$ is small, i.e., UV, then the expansion with respect to $|n|/\alpha$ is valid since we consider the situation $|t_f - t_i|\Lambda \to \infty$ and $\Lambda \sim 1$. In this case, the corresponding two-point functions are given by (4.14) and (4.33) with classical and spin angular momentum contribution $(|t_f - t_i|\Lambda)^{-\frac{4}{5-p}(J+\frac{1}{2}N^{\mathrm{f}})}$. On the other hand, the result for $|n|/\alpha \to \infty$ represents the IR property.[5]

For $p = 4$, the expansion with respect to small $|n|/\alpha$ represents the IR behavior: The resulting S-matrix is given by (4.18) and (4.35) and we see that the (normalized) two-point functions are the same as for the supergravity modes $n = 0$. If $|t_f - t_i|^{\frac{2(3-p)}{5-p}} \frac{L^4 n^2}{J^2}$ is large, i.e., UV, then $|n|/\alpha \to \infty$. In this case, the corresponding two-point functions behave like massive particles as we see from (4.49). However, the result does not mean the appearance of finite correlation length since the equation does not hold for long distance.

For $p = 3$, the expansion of two-point functions with respect to $L^4 n^2/J^2$ corresponds to the perturbative expansion of the gauge theory side since $L^4 \sim g_{\mathrm{YM}}^2 N$: At the n^2 order, this effect appears as $\sim n^2 g_{\mathrm{YM}}^2 N/J^2 \ln(|t_f - t_i|\Lambda)$ which corresponds to the explicit perturbative calculation of the gauge theory [3, 15]. On the other hand, for $p \neq 3$, the expansion with respect to $|n|/\alpha$ does not correspond to the effect of perturbative expansion since the combination $|t_f - t_i|^{\frac{2(3-p)}{5-p}} \frac{L^4 n^2}{J^2}$ is written by the original coordinates as

$$(|t_f - t_i|_{\mathrm{orig.}})^{\frac{2(3-p)}{5-p}} (g_{\mathrm{YM}}^2 N)^{\frac{2}{5-p}} \frac{n^2}{J^2} \tag{4.53}$$

and it has fractional power of the coupling constant.

In practice, it is difficult to check the results from the gauge theory side since there are severe infra-red ($p < 3$) or ultra-violet ($p = 4$) divergence. Furthermore, the result for supergravity modes already has the non-trivial form (4.3): We see that the dimension of the scalar ϕ_l does not correspond to that of free-field. For string higher modes $n \neq 0$, this fact shows that even if the quantity (4.53) is small, the first term of the corresponding expansion (i.e., (4.14) or (4.33) for $p < 3$) is different from free-propagator. For $p = 4$, the result shows that in the infra-red limit the two-point functions for any n degenerate to the one for $n = 0$ modes. This means that we have non-trivial infra-red fixed points for the corresponding $d = 5$ theory where free-fields of effective dimension $d_{\mathrm{eff}} = 6$ appear.

Also, there is a problem which operator we should identify for each sector of S-matrix including $n \neq 0$ modes. We assume that we can obtain the *diagonalized* two-point functions from the string S-matrix by our method. For $p = 3$, we know that the BMN-operators such as

$$\mathcal{O} \sim \sum_{j=1}^{J} e^{\frac{2\pi i j n}{J}} \mathrm{Tr}[Z^j \phi_a Z^{J-j} \phi_b]$$

[5]In the IR limit for $p < 3$, the (tunneling) null geodesic approaches $r \to 0$ where the dilaton expectation value diverges and we cannot neglect the string loop effect there. This corresponds to the $\frac{1}{N}$-correction in the $N \to \infty$ limit with large $g_{\mathrm{YM}}^2 N$.

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give the correct result. For $p \neq 3$, we may expect that the similar BMN-type operators play the role as the operators with diagonalized two-point functions. (For supergravity modes $n = 0$, this choice is consistent with the analysis from supergravity theories.) However, as we have stated above, it is difficult to check them in the gauge theory side.

Finally, we comment on the effect of zero-point energies from stringy modes. In the analysis of supergravity modes, the results including the effect of zero-point energies agree with the analysis from supergravity theory [2]. If the result is true, the zero-point energies from $n \neq 0$ modes must cancel for bosonic and fermionic fluctuations. For $p < 3$, the effects of zero-point energies from bosonic and fermionic fluctuations for small $|n|/\tilde{\alpha}$ can be read from (4.14) and (4.33) respectively. If this expansion is valid for large $|n|$, the corresponding zero-point energies seem to diverge. However, for $|n| \to \infty$, the zero-point energies vanish as we see from (4.46) and (4.50). Thus we may expect that the total zero-point energies for all $n \neq 0$ modes vanish and that our result is consistent with the analysis from the supergravity theory. The situation for $p = 4$ is similar.

5. Concluding remarks

We have investigated the S-matrix for superstring fluctuations around the null geodesic of the Dp-brane background ($0 \leq p \leq 4$) and have given a prediction to the two-point functions for BMN type operators in the $(p+1)$-dimensional gauge theory by assuming the holographic correspondence. In particular, we have studied the effect of string higher modes $n \neq 0$ that had remained to be analyzed in the previous papers [1, 2]. The main results for two-point functions are collected in section 4.3.

With the analysis given above, we have almost completed the analysis of two-point functions from the string theory side. The remaining task is, as we have emphasized repeatedly in our previous papers, to analyze the correlation functions in terms of the dual gauge theory itself, though it would be a non-trivial task to perform.

Our results of two-point functions from string theory for $p \neq 3$ have non-trivial properties such that the two-point functions for supergravity modes have power-law behavior with fractional power, or the effect of string higher modes $n \neq 0$ is given by the expansion of fractional powers of $g_{\rm YM}$ as in (4.53). These predictions may seem to contradict with naive expectation of gauge theory. For example, for $p < 3$, we naively expect that the two-point functions reduce to the ones for free-field in the ultra-violet limit due to the super-renormalizability of the theory. However, we must recall that we still do not have reliable result from the analysis of gauge theory in itself for $p \neq 3$: For $p < 3$, it is difficult to calculate even two-point functions including quantum effect since there is severe infra-red divergence. Our result for $n \neq 0$ represented by the expansion with respect to (4.53) (such as (4.14) or (4.33)) has structure which is different from perturbative expansion. The result cannot be rejected only because it does not correspond to the perturbative expansion: Under inclusion of non-perturbative effect, expansion is possible to take a different form. We have given a proposal for two-point functions of $(p+1)$-dimensional SYM theories from the analysis of string theory. The result is consistent with previous field theory analyses by supergravity theory, though we do not yet have concrete evidence from gauge theory. We hope that we could test our prediction by some non-perturbative analysis of the gauge theory.

Finally, since we have a definite procedure to obtain two-point functions from string S-matrix around geodesic respecting the holographic principle, it is easy to apply our method to other cases. In particular, it would be interesting to consider the non-BPS expanding strings like spinning strings [17, 18].

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A. Diagonalization of S-matrix from fermionic oscillators

We will briefly explain the relation between two expressions (3.27) and (3.31) of the fermionic part of S-matrix S_{f} and perform 'diagonalization' by T-dependent Bogoliubov transformation.

Normal-ordered form → Exponential form. In order to see the relation between $(A_n^{\mathrm{f}}, B_n^{\mathrm{f}})$ in (3.27) and $(\tilde{A}_n^{\mathrm{f}}, \tilde{B}_n^{\mathrm{f}})$ in (3.31), we calculate

$$V_n \equiv \begin{pmatrix} S_{\mathrm{f}} d_n S_{\mathrm{f}}^{-1} \\ S_{\mathrm{f}} d_n^\dagger S_{\mathrm{f}}^{-1} \end{pmatrix}$$

in both representations. The results are respectively written as

$$V_n = \frac{1}{1+B_n} \begin{pmatrix} 1 & A_n \\ A_n & (1+B_n)^2 + A_n^2 \end{pmatrix} \begin{pmatrix} d_n \\ d_n^\dagger \end{pmatrix} \tag{A.1}$$

and

$$\begin{aligned} V_n &= \exp \begin{pmatrix} -\tilde{B}_n & \tilde{A}_n \\ \tilde{A}_n & \tilde{B}_n \end{pmatrix} \begin{pmatrix} d_n \\ d_n^\dagger \end{pmatrix} \\ &= \left[\cosh \Omega_n + \begin{pmatrix} -\tilde{B}_n & \tilde{A}_n \\ \tilde{A}_n & \tilde{B}_n \end{pmatrix} \frac{\sinh \Omega_n}{\Omega_n} \right] \begin{pmatrix} d_n \\ d_n^\dagger \end{pmatrix} . \end{aligned} \tag{A.2}$$

To relate N_n and $\tilde{N}_n$, we calculate $\langle 0|S_{\mathrm{f}}|0\rangle$ in both representations and as a result we have

$$\prod_{\alpha,n} N_n^{\mathrm{f}} = \left(\prod_{\alpha,n} \tilde{N}_n^{\mathrm{f}} \right) \langle 0| \sum_{\alpha,n} \left[\frac{1}{2} \tilde{A}_n^{\mathrm{f}} d_n^{\alpha\dagger} d_{-n}^{\alpha\dagger} + \tilde{B}_n^{\mathrm{f}} d_n^{\alpha\dagger} d_{-n}^{\alpha} + \frac{1}{2} \tilde{A}_n^{\mathrm{f}} d_n^{\alpha} d_{-n}^{\alpha} \right] |0\rangle . \tag{A.3}$$

This equation can be simplified by using the following technique. We define

$$S_{\rm f}(\epsilon) \equiv \left(\prod_{\alpha,n} \tilde{N}^{\rm f}_n\right) \exp\left\{\epsilon \sum_{\alpha,n} \left[\frac{1}{2}\tilde{A}^{\rm f}_n d^{\alpha\dagger}_n d^{\alpha\dagger}_{-n} + \tilde{B}^{\rm f}_n d^{\alpha\dagger}_n d^{\alpha}_{-n} + \frac{1}{2}\tilde{A}^{\rm f}_n d^{\alpha}_n d^{\alpha}_{-n}\right]\right\} \tag{A.4}$$

and differentiate $\langle 0|S_{\rm f}(\epsilon)|0\rangle$ with respect to ϵ:

$$\frac{d}{d\epsilon}\langle 0|S_{\rm f}(\epsilon)|0\rangle = \left(\prod_{\alpha,n}\tilde{N}^{\rm f}_n\right)\langle 0|S_{\rm f}(\epsilon)\sum_{\alpha,n}\frac{1}{2}\tilde{A}^{\rm f}_n d^{\alpha\dagger}_n d^{\alpha\dagger}_{-n}|0\rangle\,. \tag{A.5}$$

By using the relation corresponding to (A.2) with ϵ, the right-hand side is rewritten as the form [coefficients] $\times$ $\langle 0|S_{\rm f}(\epsilon)|0\rangle$ and we can solve (A.5) as a differential equation of ϵ. The result is

$$\langle 0|S_{\rm f}(\epsilon=1)|0\rangle = \prod_{\alpha,n}\left(\tilde{N}^{\rm f}_n \exp\left(\frac{\tilde{B}^{\rm f}_n}{2}\right)\frac{1}{\sqrt{1+B^{\rm f}_n}}\right). \tag{A.6}$$

In fact we can show that

$$\tilde{N}^{\rm f}_n = N^{\rm f}_n \exp\left(-\frac{\tilde{B}^{\rm f}_n}{2}\right)\sqrt{1+B^{\rm f}_n}\,. \tag{A.7}$$

Diagonalization. We can diagonalize $S_{\rm f}$ in the exponential representation as

$$\begin{aligned} S_{\rm f} &= S_{{\rm f},n=0}\prod_{\alpha,n>0}(\tilde{N}^{\rm f}_n)^2 \exp\left[\tilde{A}^{\rm f}_n d^{\alpha\dagger}_n d^{\alpha\dagger}_{-n} + \tilde{B}^{\rm f}_n (d^{\alpha\dagger}_n d^{\alpha}_{-n} + d^{\alpha\dagger}_{-n} d^{\alpha}_n) + \tilde{A}^{\rm f}_n d^{\alpha}_n d^{\alpha}_{-n}\right] \\ &= S_{{\rm f},n=0}\prod_{\alpha,n>0}(\tilde{N}^{\rm f}_n)^2 \exp(\tilde{B}^{\rm f}_n + \Omega^{\rm f}_n)\exp\left[-\Omega^{\rm f}_n\left(c^{\alpha\dagger}_n(T)c^{\alpha}_{-n}(T) + c^{\alpha\dagger}_{-n}(T)c^{\alpha}_n(T)\right)\right]. \end{aligned} \tag{A.8}$$

We use T-dependent operators $(c^{\alpha\dagger}_n(T), c^{\alpha}_{-n}(T))$ satisfying (3.38) defined by

$$\begin{pmatrix} c^\dagger_n(T) \\ c_n(T)\end{pmatrix} = \begin{pmatrix} G^{\rm f}_n(T) & F^{\rm f}_n(T) \\ E^{\rm f}_n(T) & D^{\rm f}_n(T)\end{pmatrix}\begin{pmatrix} d^\dagger_n \\ d_n\end{pmatrix}. \tag{A.9}$$

Here $D^{\rm f}_n$, $G^{\rm f}_n$, $F^{\rm f}_n$ and $E^{\rm f}_n$ are determined in order to satisfy the relations $D^{\rm f}_n G^{\rm f}_n - F^{\rm f}_n E^{\rm f}_n = 1$, $(D^{\rm f}_n, G^{\rm f}_n, F^{\rm f}_n, E^{\rm f}_n) = (D^{\rm f}_{-n}, G^{\rm f}_{-n}, -F^{\rm f}_{-n}, -E^{\rm f}_{-n})$ and

$$\begin{pmatrix} D^{\rm f}_n & \mp E^{\rm f}_n \\ \mp F^{\rm f}_n & G^{\rm f}_n\end{pmatrix}\begin{pmatrix} \pm\tilde{A}^{\rm f}_n & \tilde{B}^{\rm f}_n \\ -\tilde{B}^{\rm f}_n & \pm\tilde{A}^{\rm f}_n\end{pmatrix}\begin{pmatrix} D^{\rm f}_n & \pm F^{\rm f}_n \\ \pm E^{\rm f}_n & G^{\rm f}_n\end{pmatrix} = \begin{pmatrix} 0 & -\Omega^{\rm f}_n \\ \Omega^{\rm f}_n & 0\end{pmatrix}. \tag{A.10}$$

If necessary, we can diagonalize $S_{{\rm f},n=0}$ separately, though it is already diagonalized in our case. Thus, the final form of diagonalized S-matrix becomes (3.37).

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